Laboratory I.5
Relationship between a Function and Its Derivative

Goal

• Given the graph of a function, to be able to visualize the graph of its derivative.

Before the Lab

In this laboratory, you will be asked to compare the graph of a function like the one in Figure 1 to that of its derivative. This exercise will develop your understanding of the geometric information that $f''$ carries.

![Graph of $f(x)$ and $f''(x)$](image)

Figure 1: $f(x) = x^2(x - 1)(x + 1)(x + 2)$

You will need to bring an example of such a function into the lab with you, one whose graph meets the $x$-axis at four or five places over the interval $[-3, 2]$ (see "Ready for Lab?"). During the lab, your partner will be asked to look at the graph of your function and describe the shape of its derivative (and you will be asked to do the same for your partner’s function). One way to make such a function is to write a polynomial in its factored form. For example, $f(x) = x^2(x - 1)(x + 1)(x + 2)$ is the factored form of the function in Figure 1. Its zeros are at 0 (twice), 1, -1, and -2.

In the Lab

1. Author $f(x) := x^2(x^2 - 1)(x + 2)$.

a. Find the derivative of $f$. Plot the graphs of both $f$ and $f''$ in the same viewing rectangle over the interval $[-2.5, 1.5]$. 
Answer the following questions in interval notation by inspection of this graph in the spaces below, and turn this sheet in with your lab:

b. Over what intervals does the graph of \( f \) appear to be rising as you move from left to right?

c. Over what intervals does the graph of \( f' \) appear to be above the \( x \)-axis?

d. Over what intervals does the graph of \( f \) appear to be falling as you move from left to right?

e. Over what intervals does the graph of \( f' \) appear to be below the \( x \)-axis?

f. What are the \( x \)-coordinates of all of the peaks and valleys of the graph of \( f \)? Identify which is a peak and which is a valley.

g. For what values of \( x \) does the graph of \( f' \) appear to meet the \( x \)-axis?

2. **Author** \( f(x) := \frac{x}{1 + x^2} \).

a. Find the derivative of \( f \). Plot the graphs of both \( f \) and \( f' \) in the same viewing rectangle over the interval \([-3, 3]\).

Now answer the same set of questions as in parts b-g above:

b. Over what intervals does the graph of \( f \) appear to be rising as you move from left to right?

c. Over what intervals does the graph of \( f' \) appear to be above the \( x \)-axis?

d. Over what intervals does the graph of \( f \) appear to be falling as you move from left to right?

e. Over what intervals does the graph of \( f' \) appear to be below the \( x \)-axis?

f. What are the \( x \)-coordinates of all of the peaks and valleys of the graph of \( f \)?

g. For what values of \( x \) does the graph of \( f' \) appear to meet the \( x \)-axis?

3. On the basis of your experience so far, write a statement (or statements) about properties you have observed about the graph of its derivative.

a. How is the graph of the derivative when a function is rising?

b. How is the graph of the derivative when a function is falling?

c. What happens to the graph of the derivative when a function has a peak or a valley?

4. Now let \( g \) be the function that your lab partner brought into the lab. (If you have no partner, just use your own function.) In this problem you will use your statement from part 3 above, to predict the shape of the graph of \( g' \), given only the shape of the graph of \( g \).
a. Have your lab partner produce a plot of the graph of $g$ over the interval $[-3, 2]$. Your partner may need to adjust the vertical scale of the window to capture all of the action. Print this graph. On the basis of this plot, use your conjecture to imagine the shape of the graph of $g'$. In particular, find where $g'$ is above, where $g'$ is below, and where $g'$ meets the $x$-axis. Carefully sketch a graph of $g'$ on your printout of $g$. The basic shape is more important than the completely correct graph.

b. Now have your lab partner add a plot of the graph of $g'$ to the Derive plot of $g$, and print the result. How did you do?

c. Reverse roles with your lab partner and do parts a and b again.

5. Author and Plot in a new Plot window $f(x) = |x^2 - 4|$.

a. There are two values of $x$ for which the derivative does not exist. What are these values, and why does the derivative not exist there?

b. Find the derivative of $f$ at those values of $x$ where it exists, and add it to the Plot window.

c. Look at your plot. Does your conjecture from Problem 4 still hold? Do you need to make any modifications? If so, make your additions or corrections below.

Ready for Lab?

1. Give another example of a polynomial $g$ of degree at least 5 (in factored form is OK) with four or five real zeros between -3 and 2. You will use this polynomial in Problem 4.

a. Your polynomial $g(x) =$ ___________________________.

b. Its zeros: ___________________________

c. Its derivative $g'(x) =$ ___________________________. If your function is complicated, you may want to use the computer to calculate its derivative.

After the Lab

There is no separate "After the Lab" section; turn in the work in the blanks above and the graphs you printed.
Extra Credit Problems

1. Consider the function \( f(x) = 2^x \). Some people think that \( f''(x) = x 2^{x-1} \). Graph \( f(x) \) from \( x = -4 \) to 4, and based on the graph and your conjecture in part 3 of “In The Lab”, explain why this cannot be true.

2. This laboratory has given you experience in using what you know about the shape of the graph of a function \( f \) to visualize the shape of its derivative function \( f'' \). What about going backwards? Suppose that your partner had given you the graph of \( f'' \), would you be able to reconstruct the shape of the graph of \( f \)? If \( f'' \) is positive, for example, does your conjecture enable you to rule out certain possibilities for the shape of \( f \)? The graph in Figure 2 is a sketch of the derivative of \( f \). Use your conjecture to construct a possible graph for the function \( f \) itself. The important part of this problem is neither the actual shape that you come up with, nor its position in the \( xy \)-plane, but your reasons for choosing it. Why isn't there a unique function that has \( f'' \) for its derivative?

Figure 2: The graph of \( y = f'(x) \)
3. Figure 3 shows the graphs of three functions. One is the position of a car at time $t$ minutes, one is the velocity of that car, and one is its acceleration. Identify which graph represents which function and explain your reasoning.

Figure 3: Position, velocity and acceleration graphs.