Laboratory I.6
Investigating the Intermediate Value Theorem and Fixed Points

Goals

• The student will discover and acquire a feel for one of the major theorems in calculus.
• The student will apply the theorem in practical and theoretical ways.
• The student will understand why continuity is required for the theorem.

Before the Lab

The Intermediate Value Theorem (IVT) is an important theorem in later mathematics. (A theorem is a mathematical fact which have been proved true.) We’re going to have you work some examples related to the theorem, and you ought to be able to figure out what the theorem says by thinking about the examples. (Frankly, this is one of those “deep insights” which often inspires reactions like “Well, duh, of COURSE it’s true, what’s the big deal?” As you go on in math, though, you’ll find more and more “obvious” things that are hard to prove true.) Anyhow, here all we’re asking you to do is discover the IVT.

We’re going to spend more time here in “Before the Lab” talking about fixed points, which turn out to have important significance in both theory and practice. Some of the really slick methods computers use to solve equations depend on fixed points. We want to define them, and help you see them.

Let \( f(x) = (x - 1)^2 \). When you plug in \( x = \frac{3 - \sqrt{5}}{2} \), you find that

\[ f\left( \frac{3 - \sqrt{5}}{2} \right) = \frac{3 - \sqrt{5}}{2}. \]

That is, you get the same number back that you plugged in. This is what a fixed point is: a fixed point for a function \( f(x) \) is some number \( x = c \) such that \( f(c) = c \).

What do fixed points look like? Here’s a graph of the example above, as well as the line \( y = x \).
is the x-value between 0 and 1 where the two graphs cross. Why? Well, what would you do if we asked you to find that point of intersection? You’d solve the equation \((x - 1)^2 = x\). But then that’s \(f(x) = x\), which would be a fixed point. So, fixed points can be seen as the intersection of the graphs of \(f(x)\) and \(y = x\).

In the Lab

1. **Author** \(f(x) := x^5 - 4x^2 + 1\). We’re going to consider how it behaves between \(x = 1\) and \(x = 2\).
   
   a. Compute the numerical values of \(f(1)\) and \(f(2)\). Based on that, think about why the graph of \(f\) ought to cross the x-axis somewhere between \(x = 1\) and \(x = 2\). That is, why must there be a number \(c\) between 1 and 2 such that \(f(c) = 0\)?

   b. **Plot** \(f(x)\) to confirm that its graph indeed does cross the x-axis. Estimate the value \(c\) where it does. Print this graph, and mark \(c\) and give its value on the graph.

2. Here are two more examples.
   
   a. Let \(g(x) = \begin{cases} 
   x^2 + 1, & \text{for } x \leq 0 \\
   1 - x^2 - x^4, & \text{for } x > 0 
   \end{cases}\) In Derive, you would **Author**

   \(g(x) := \text{if}(x \leq 0, x^2 + 1, 1 - x^2 - x^4)\). Compute \(g(-3)\) and \(g(2)\), **Plot** \(g(x)\), and find the point(s) \(c\) where \(g(x)\) intersects the x-axis. Print this graph, and mark \(c\) and give its value on the graph.

   b. Finally, let \(h(x) = \begin{cases} 
   x^2 + 1, & \text{for } x \leq 0 \\
   -1 - x^2 - x^4, & \text{for } x > 0 
   \end{cases}\). Compute \(h(-3)\) and \(h(2)\), **Plot** \(h(x)\), and notice you can’t find any point(s) \(c\) where \(h(x)\) intersects the x-axis. Print this graph.

   c. Think about what examples 1, 2a and 2b had in common, and what was different. (Hint: what can you say about whether the functions are continuous or not?) Then go ahead and
do “After the Lab” #1 (fill in the blanks). This is the Intermediate Value Theorem.

3. **Author** the function \( k(x) = x^3 - e^x + e^{-x} \) and evaluate \( k(-5) \) and \( k(5) \). What does the IVT say will happen? Now **Plot** \( k(x) \), and notice that, between \(-5 \) and \( 5 \), it happens five times. Find six \( x \) values for this function so that the sign of \( k(x) \) alternates \(+---\) (or \(-+--\)) for those six points. Print your graph and mark the points. The point of this example is that the IVT guarantees only that the function will cross the \( x \)-axis, but doesn’t predict how many times.

4. **Author** and **Plot** the following functions. Also, for each plot also add the function \( y = x \). Your goal is to understand when a function has a fixed point between some point \( x = a \) and \( x = b \). (You don’t have to print the plots, we’re running out of forests for the paper, but jot notes on the results for “After the Lab” #6.)

   a. \( y = x^3 \) between \( x = 0.5 \) and \( x = 1.5 \).
   b. \( y = x^3 \) between \( x = 2 \) and \( x = 3 \).
   c. \( y = \cos x \) between \( x = 0 \) and \( x = \pi \).
   d. \( y = \text{round}(1 - x) \) between \( x = 0 \) and \( x = 1 \). (The round function in Derive just rounds off the number. Before you do this, you must use **File Load Math . . .** to read in the file Number.mth.)

Ready for Lab?

1. Define what it means for a function to be continuous.

2. Define a fixed point.

3. Find the other fixed point in the example from “Before the Lab”.


After the Lab

1. Based on your “In the Lab” work on #1 and #2, fill in the blanks in the following statement of the Intermediate Value Theorem:

Given a ________________ function $f$ defined on the closed interval $[a, b]$ such that $y = 0$ is between ________________ and ________________, there is some point $c$ between ________________ and ________________ such that ________________.

2. You are in your thermostat-controlled 78˚ home. If your oven is 250˚ and you turn it off, is there ever an instant when the oven temperature is 170˚? Explain your answer and its relationship to the IVT.

3. If you remove marbles from a bag one at a time, must there always come a time when the bag contains exactly half the number of marbles it began with? Again, explain your answer, and its relationship to the IVT.
4. In the space below, sketch the graphs of two generic continuous functions \( g \) and \( h \) from \( x = 0 \) to \( x = 1 \), with the property that \( h(0) < g(0) \) and \( h(1) > g(1) \). Do these two functions cross? Must they always cross? (Hint: what does the IVT say about the function \( g - h \)?)

5. One plate has been in the freezer for a while, and another has been in a warm oven. Then you switch the locations of the plates. Will there be a moment when the plates have exactly the same temperature? What does this have to do with anything above?

6. Using the examples from “In the Lab” #4, describe in your own words what conditions a function has to satisfy to have a fixed point.

7. (Extra credit) Show that on any great circle around the earth there are always two points at opposite ends of a diameter which have the same temperature. (A great circle is a circle whose center is also the center of the earth. An example is the equator.)