MOTIVATIONAL PROBLEM FOR SETS

A US shuttle is to be sent to a space station in orbit around the earth and 700 kilograms of its payload are allotted to experiments designed by scientists. Researchers from around the country apply for the inclusion of their experiments. Of course, they must specify how much equipment they want taken into orbit will weigh. A panel of reviewers then decides which proposals are reasonable. These proposals are rated from 1 (the lowest score) to 10 (the highest) on their potential importance to science. The ratings are listed below.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Weight in Kg.</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Cloud patterns</td>
<td>36</td>
<td>5</td>
</tr>
<tr>
<td>2 Speed of light</td>
<td>264</td>
<td>9</td>
</tr>
<tr>
<td>3 Solar power</td>
<td>188</td>
<td>6</td>
</tr>
<tr>
<td>4 Binary stars</td>
<td>203</td>
<td>8</td>
</tr>
<tr>
<td>5 Relativity</td>
<td>104</td>
<td>8</td>
</tr>
<tr>
<td>6 Seed viability</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>7 Sun spots</td>
<td>92</td>
<td>2</td>
</tr>
<tr>
<td>8 Mice tumors</td>
<td>65</td>
<td>8</td>
</tr>
<tr>
<td>9 Weightless vines</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>10 Space dust</td>
<td>170</td>
<td>6</td>
</tr>
<tr>
<td>11 Cosmic rays</td>
<td>80</td>
<td>7</td>
</tr>
<tr>
<td>12 Yeast fermentation</td>
<td>22</td>
<td>4</td>
</tr>
</tbody>
</table>

It is decided to choose experiments so that the total of all their ratings is as large as possible. Since there is also the limitation that the total weight cannot exceed 700 kilograms, it is not clear how to do this. If we just start down the list, for example, experiments 1, 2, 3 and 4 have a total weight of 691 kilograms, and total rating of 28. Now we cannot take experiment 5, since its 104 kilograms would put us over the 700-kilogram limit. We could include experiment 6, however, which would bring us up to 698 kilograms, and the rating would be 34. So far we have shown one possible choice of experiments with a total weight of 698 and rating of 34.

EXERCISE 1

1. Is there a better choice to make? Why do you think it is a better one? Can you improve this one?

2. A possible approach to solve this problem would be to consider all possible collections of experiments. For instance experiments 2 and 10, or 3, or 1, 5, 7, 11, 12, etc. and for each choice determine the total weigh. Hence, if the total weigh is less than or equal to 700 kg the collection would be a candidate to be taken. How many cases do we need to consider under this approach? Can you think of a way to determine the total number of choices? A “good” guess would be okay at this stage.

We will find a way to answer the latter question using the concept of a set.
SETS
A well-defined collection of objects is called a set. What we mean by well-defined is that whenever an object is given, it can be determined precisely whether or not the object is in the collection.
In our example, the collection of all experiments can be considered as a set. Let’s call it the set \( G \). Instead of writing the names of the experiments, we label them from 1 through 12. Hence, we have:
\[
G = \{1,2,3,4,5,6,7,8,9,10,11,12\}
\]
In this case 1 is in \( G \). We say that 1 is an element of \( G \), or that 1 belongs to \( G \). It is denoted as \( 1 \in G \). Likewise, \( 2 \in G \), \( 3 \in G \), ..., \( 12 \in G \). However, 0 is not an element of \( G \). It is denoted by \( 0 \notin G \). Also, \( 13 \notin G \) and milk \( \notin G \).

NOTATION
Sets are denoted by capital letters, the elements of a set are enclosed by curly brackets ({}), and commas separate elements.
\( A=\{1,2,3\} \) represents a set with three elements, \( B=\{12,3\} \) represents a set with two elements. \( A = 1,2,3 \) does not represent a set. Do not overlook the commas when writing sets.

It is very important for us to count the number of elements of a set, as we will see later. To indicate the number of elements of a set the name of the set is enclosed within bars. For example the number of elements of the set \( G \) is denoted by \( |G| \). In this case \( |G|=12 \). Also, for \( A \) and \( B \) above, \( |A|=3 \), and \( |B|=2 \).
When a set has \( n \) elements, for \( n \) a positive whole number, the set is called a finite set. All the examples we have so far are of finite sets. If a set is not finite, it is called an infinite set.
Examples of infinite set are:
\[
\begin{align*}
M &= \{2, 4, 6, 8, 10, \ldots\} \\
S &= \{\ldots-3, -1, 1, 3, 5, \ldots\} \\
T &= \{a, ab, aba, abab, \ldots\}
\end{align*}
\]

EXCERSICE 2
1. Consider the set \( G = \{1,2,3,4,5,6,7,8,9,10,11,12\} \). Determine the truth value of the following statements:
\[
3 \in G, \quad \{1\} \in G, \quad 7 \notin G, \quad \{10\} \notin G
\]

2. Let \( M=\{01,1,0,11\} \), \( N=\{\{1\}, 0\} \), and \( P=\{\text{Students in the Discrete Math class at TAMU-CC this semester}\} \)
   a) For each of these sets, write two elements in the set and two elements not in the set (use the proper notation)

   b) Complete \( |M|= \quad |N|= \quad |P|= \)

Sometimes a set can be described by a property satisfied by its elements. For example the set
\[
K=\{0, 2, 4, 6\} \]
is formed by natural numbers, which are even natural numbers less than 8. We write it as

\[ K = \{ x \mid x \text{ is a non-negative even whole number less than 8} \} \]

Of course there are other ways to represent the same set. Here are some other ways to represent the set \( K \):

\[ K = \{ x \mid x \text{ is an even whole number less than 7} \} \]

\[ K = \{ x \mid x \text{ is an even whole number less than or equal to 6} \} \]

\[ K = \{ x \mid x \text{ is an even whole number greater than or equal to zero and less than 8} \}. \]

Before we get deeper into sets, let’s mention some sets we will refer to with some frequency:

- The set of natural numbers, \( \mathbb{N} \)

\[ \mathbb{N} = \{0, 1, 2, 3, 4, \ldots \} \]

- The set of integer numbers, \( \mathbb{Z} \)

\[ \mathbb{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

- The set of rational numbers, \( \mathbb{Q} \). This is the set of all possible ratios of integer numbers, excluding zero as a denominator.

\[ \mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\} \]

With this notation handy, we can represent the set \( K \) above in a more technical way:

\[ K = \{ x \mid x = 2q, q \in \mathbb{Z}, 0 \leq q \leq 3 \} \]

Note that each value of \( q \), in its range, will produce an element of \( K \). Also, observe that in this case we can use \( q \in \mathbb{Z} \) or \( q \in \mathbb{N} \) and there is no change in the set.

**EXERCISE 3**

Somebody suggested representing the set \( K \) as

\[ K = \{ x \mid x = 2q, q \in \mathbb{Z} \} \text{ or } K = \{ x \mid x = 2q, q \in \mathbb{Z}, 0 < q \leq 3 \}? \]
What is wrong with these representations for K?

Let’s describe other sets using a property that is satisfied only by their elements:

The set \( T = \{-2, -1, 0, 1, 2, 3\} \) can be written as

\[
T = \{ x \mid x \text{ is an integer, } -3 < x < 4 \}
\]

\[
T = \{ x \mid x \text{ is an integer, } -2 \leq x < 4 \}
\]

\[
T = \{ k \mid k \in \mathbb{Z}, -2 \leq k \leq 3 \}
\]

Notice that when a set is described by a property satisfied by its elements, the property must describe only the elements in the set. For example, the elements in the set \( K \) above are not completely described by the property: “The elements of \( K \) are even numbers”. This is because, 8 is an even number but 8 is not in \( K \). Likewise, the property: “The elements in \( K \) are the natural numbers between 0 and 6” does not describe the elements in \( K \) since 5 satisfies the property but is not in \( K \).

**EXERCISE 4**

1. Consider the set

\[
M = \{2, 4, 6, 8, 10\}
\]

Explain why the following sets do not represent \( M \):

   a. \( \{ x \mid x \text{ is an even number} \} \)

   b. \( \{ x \mid x \text{ is an even natural number} \} \)

   c. \( \{ x \mid x \text{ is an even number less than or equal to 10} \} \)

2. Write the set \( M \) by describing a property satisfied by its elements.

3. Write each of the sets below by using a property satisfied by their elements:

   a. \( \{ b, d, f, h \} \)

   b. \( \{-3, 0, 3, 6, 9, 12, \ldots\} \)

   c. \( \{3.45, 3.4545, 3.454545, \ldots\} \)
4. Consider the set \( U = \{0, \{0\}, 2, \{0,1,2\}\} \)

   a. List the elements of \( U \)

   b. What is \(|U|\)

Sometimes sets are represented using Venn diagrams (Introduced by John Venn, 1834-1923, to illustrate relationships between sets). For instance the Venn diagram for the set \( M = \{0,1,10,11\} \) is

![Venn Diagram for M](image)

In the choices of experiments for the shuttle, each choice of experiments corresponds to a set, and each element in the chosen set is an element of \( E \), the set of all the experiments. This leads to our next topic, subsets

**SUBSET**

Let \( A \) and \( B \) be any two sets. If each element of \( A \) is also an element of \( B \), we say that \( A \) is a subset of \( B \), or \( A \) is contained in \( B \).

Let \( B = \{1,2\} \), \( A = \{1\} \), \( C = \{2\} \), \( D = \{1,2\} \), \( R = \{2,1\} \), and \( F = \{1,3\} \).

- C, D, and R are subsets of B. Why?
- The set D is not a subset of F, because we can show one element of D that is not an element of F. Which one?
- The sets R and D satisfying that \( R \subseteq D \), and \( D \subseteq R \). Actually we say that \( R = D \)
- F is not a subset of B because \( 3 \in F \) but \( 3 \notin B \). We denote that F is not a subset of B by \( F \not\subseteq B \).

The Venn diagram to represent the relationship \( A \subseteq B \) for any two sets is

![Venn Diagram for Subset](image)
This diagram considers all the possibilities for. These are:

- A and B are distinct, but \( A \subseteq B \). It means that every element in A is in B, but B has elements which A are not in A. Sometimes it is necessary to emphasize this situation. When \( A \subseteq B \), but A and B are not equal (\( A \neq B \)), we say that A is a **proper subset** of B. This is denoted \( A \subset B \).
- A and B have the same elements. In this case \( A \subseteq B \) and \( B \subseteq A \) and we say that \( A = B \).

One more comment with regard to notation. \( A \subseteq B \) indicates that either \( A = B \) or \( A \subset B \).

**EXERCISE 5**

1. Give an example of sets A and B where \( A \mid B \) but \( A \neq B \). List the elements of the sets and draw the Venn diagram for the sets in standard position.
2. What is the condition to be satisfied to guarantee that a set A is not a subset of a set B? State it in English. Illustrate your statement with two examples.

**NULL SET**

Consider the collection of all students in the Discrete Math class this semester who have finished High School. As you can see this collection has no elements. The set with no elements is called the **Null set or Empty set**. This is denoted by \( \emptyset \) or \{ \}.

**EXERCISE 6**

\( \checkmark \) Let T be any set. One of the two statements below is true:

- The empty set is a subset of T
- The empty set is not a subset of T.

Determine which one is true. Explain your reasoning for the one that is true. Give an example for the one, which is false.

**FACT**

The null set is a subset of any set.

Given a set M, the collection of all subsets of M, is called the **Power Set of the set M**. This is denoted \( \mathcal{P}(M) \).

If \( A = \{1,2\} \), \( \mathcal{P}(A) = \{\emptyset,\{1\},\{2\},\{1,2\}\}\)

Notice that the power set is a set of sets, or a set whose elements are set. A set like this usually called a family of sets.
EXERCISE 7

1. Consider the sets $\mathcal{P}(\emptyset)$, $\mathcal{P}({1})$, $\mathcal{P}({1,a})$, and $\mathcal{P}({a,b,c})$.
   
   a) Write all the elements for each of these sets.
   
   b) Indicate the number of elements of each of these sets (use proper notation).

2. Calculate $|\mathcal{P}({1,2})|$ and $|\mathcal{P}({a, \{a\}})|$. Are there the same?

3. In part (1) you determined the number of elements of the power set of a set with 0, 1, 2 and 3 elements. If you look at these numbers carefully, you will see that there is a pattern to determine the number of elements of the power set from the number of elements of the set.

   a) What is the pattern?

   b) Use the pattern to find the number of elements of the power set of a set with four elements. Exhibit a set with four elements and list the elements of its power set to verify your claim.

   c) Use the pattern you found in (a) to calculate the number of elements of a set with 20 elements. Do not attempt to list all the elements in the set, but assume that if you wanted to do so, you could write one element per second. How long will it take to write all the elements in the power set?

CONJECTURE

The power set of a set with $n$ elements has ______ elements.

You will find a way to justify the validity of this conjecture by using the concept of strings in the next section.

There are situations where there is the need to combine sets to obtain a new set. For example, in the case that we have chosen a set with two experiments for the shuttle trip and a set with five experiments, we may want to consider the set formed by the experiments common to both sets or the set formed by the distinct experiments between the two choices, or the set of experiments in one set that are in the other set. In each of these cases the elements form another set.

OPERATIONS BETWEEN SETS

INTERSECTION

The intersection between two sets, $A$ and $B$, is the set formed by the common elements between the two sets. That is each element in the intersection of $A$ and $B$ is an element of $A$ and an element of $B$. The intersection of the sets $A$ and $B$ is denoted $A \cap B$
For example, if $A = \{1, 5, 3\}$, $B = \{2, 4, 0\}$, $C = \{0, 4, 2, 5\}$ we have:

\begin{align*}
B \cap C &= \{2, 4, 0\} \\
A \cap C &= \{5\} \\
A \cap B &= \emptyset
\end{align*}

This operation can be displayed using Venn diagrams as shown below. The intersection set is shaded and its elements are indicated in bold face.
COMMENTS

- If the intersection of two sets is the empty set, as in the case of A and B in the previous example, the sets are called **disjoint**. Therefore, saying that two sets are disjoint and that they have empty intersection is exactly the same.

- The Venn diagram of two sets in standard position is shown in the picture below. This is called in standard position because all possible relationships between the two sets are contemplated in the diagram.

  ![Venn Diagram](image)

- if the sets are disjoint, the area corresponding to the intersection of the two sets in the diagram of the sets in standard position remains unshaded or blank. They have empty intersection.

- If A is a subset of B, all the elements in the diagram will be included inside B. The intersection will be all the elements in A.

**EXERCISE 8**

1. Draw the Venn diagram for any two sets A and B, such that A \(\subseteq\) B. What can you say about A \(\cap\) B? Verify your result with an example.

2. Let A be any set. What can you say about A \(\cap\) \(\emptyset\)? Explain.

3. Let A= \{x | x is a 3 - card hand from a 52 - card deck\}, B= \{x | x is a 4 - card hand from a 52 - card deck\}. Assume that the cards are drawn from the same deck. However, the hands can be drawn at different times, and it is allowed to return the card to the deck before you draw the next hand. Notice that these sets correspond to family of sets.
   a. Exhibit one element of A and one element of B.
   b. Describe the property that characterizes the elements of A \(\cap\) B? What is this set?
   c. Show one element of A, and one of B, whose intersection is not empty.
   d. Show two elements of B, which are disjoint.

4. Let M=\{x | x is an even integer\}, N=\{x | x is a positive multiple of 4\}, P=\{x | x is a natural number multiple of 3\}. Find the following sets:
   a) M
   b) N
   c) P
   d) M \(\cap\) N
   e) N \(\cap\) P
   f) P \(\cap\) M
UNION
The union between two sets A and B is the set formed by the elements, which are in either A, or B. That is, the union is made up of elements, which are found in either A, or B. This set is denoted as \( A \cup B \).

For example, if \( A=\{1, 5, 3\} \), \( B=\{2, 4, 0\} \), \( C=\{0, 4, 2, 5\} \) we have:

- \( B \cup C = \{0, 2, 4, 5\} \)
- \( A \cup C = \{0, 1, 2, 3, 4, 5\} \)
- \( A \cup B = \{1, 2, 3, 4, 5, 0\} = \{0, 1, 2, 3, 4, 5\} \)

The operation can be displayed using Venn diagrams. The elements in the union are shown in bold face in the graph below.

EXERCISE 9
1. Draw the Venn diagram for sets A and B, such that \( A \subseteq B \). What can you say about \( A \cup B \)?
2. If A is any set, what can you say about \( A \cup \emptyset \)? Explain.
3. Let \( A=\{x \mid x \text{ is a 3 - card hand from a 52 - card deck}\} \), \( B=\{x \mid x \text{ is a 4 - card hand from a 52 -card deck}\} \). Describe \( A \cup B \) by giving a property of its elements.
4. Let \( M=\{x \mid x \text{ is an even integer}\} \), \( N=\{x \mid x \text{ is a positive multiple of 4}\} \), \( P=\{x \mid x \text{ is a natural number multiple of 3}\} \). Find the sets:
   a) \( M \cup N \)
   b) \( N \cup P \)
c) \( P \cup M \)
d) \( (M \cap P) \cup N \)

5. For any two sets \( A \) and \( B \), which of the set \( A \cup B \) or \( A \cap B \) has more elements?. Consider all the possibilities.

**REFERENCE SET**

There are many instances when we need to have a reference set in order to talk about other sets. The following examples illustrate what we call a **reference set** or **universal set**.

- The set \( S = \{0, 1, 2, 3, \ldots, 100\} \) can be taken as the reference set if we are talking about the even natural numbers less than 100.
- The set of all 3-card hands from a 52-card deck is a reference set when we talk about 3-card hands having one club.
- In a party, all the possible ways to match two people can be considered the reference set if we talk about pairs of people in the party who are either friends or enemies.

In general there is not a unique reference set. For instance in this first example above, another reference set could be \( S = \{0, 2, 4, \ldots, 102\} \). In many instances we have to decide which set to take as a reference set. Try to make your choice so that your life is not so hard!!! Sometimes we consider the smallest reference set (smallest in the sense that it is the one with the least possible number of elements).

One thing that may be clear at this point is that the reference set must contain all the sets participating in the discussion. That is, if \( A, B, C \) are sets and \( S \) is a reference set, the Venn-diagram to represent this situation looks like the one below where \( A, B, C \) are in standard position.

**DIFFERENCE**

The difference between the set \( A \) and \( B \) is the set formed by the elements which are in \( A \) but not in \( B \). This is denoted as \( A - B \).
For example, if $A=\{1, 2, 3\}$, $B=\{2, 4, 0\}$, $C=\{0, 4, 2, 5\}$ we have:

\[
\begin{align*}
A - B &= \{1, 3\} \\
B - A &= \{0, 4\} \\
A - C &= \{1, 3\} \\
C - B &= \{5\} \\
B - C &= \emptyset
\end{align*}
\]

The Venn diagram below represents the standard form for both $A-B$ and $B-A$.

In the case where $S$ is the reference set and $A$ is one of the sets participating in the discussion, the difference set $S - A$ is called the complement of $A$ with respect to $S$. Other notations for the complement of $A$ are $\overline{A}$ or $A'$. The Venn diagram to represent the complement of the set $A$ is shown in the next graph.

**EXERCISE 10**

1. Let $A=\{1, 2, 3\}$, $B=\{2, 4, 0\}$ and $S=\{0, 1, 2, 3, 4, 5\}$. Write the elements for each of the following sets.
   a) $A' =$
   b) $(A \cap B)' =$
   c) $(B - A)' =$
   d) $S' =$
   e) $B' \cup (A'-B)=
   f) \emptyset' =$
g) \( B \cap B' = \)

h) \( A \cup A' = \)

2. The shaded parts of the given graphs below can be represented using operations between sets. Indicate one such operation for each of the Venn diagrams.

![Venn Diagrams]

3. Consider the set \( A = \{ x \mid x \text{ is a 3-card hand from a } 52\text{-card deck}\} \), \( B = \{ x \mid x \text{ is a 5-card hand from a } 52\text{-card deck}\} \). Take as \( S = \{ x \mid x \text{ is a hand with less than 8 cards from a } 52\text{-card deck}\} \). Describe each of the sets below using a property satisfied by its elements:

a) \( A' \)
b) \( A \cap B \)
c) \( A \cup B \)
d) \( B' \cap A \)
e) \( A - B \)
f) \( B - A \)

4. For each of the problems below, draw sets in standard position. Some of the propositions below are true all the time for any sets. Others are true for some sets but not for all. In each case use Venn diagrams to determine the truth value of each proposition. Observe that you need to draw two diagrams, one for the left-hand side and one for the right-hand side of the statement. When the statement is true for any sets, verify the result by producing an example. When the Venn diagram shows that the statement is not always true, produce an example to illustrate sets that make the statement false.

a) \( A - B \neq B - A \)
b) \( (A \cup B)' \neq A' \cap B' \)
c) \( (A \cup B)' = (A \cap B)' \)
d) \( (A \cap B)' = A' \cup B' \)
e) \( (A \cap B)' = A' \cap B' \)
f) \( (A')' = A \)
g) \( \emptyset = S' \)
h) \( A \cup B \subseteq A \cap B \)
i) \( A \cap B \subseteq A \cup B \)
**EXERCISE 11**

Each of the following situations can be represented using sets and operations between them. In each case indicate the sets in consideration, the operation that will represent the situation, and finally indicate a choice for a reference set \( S \).

1. The natural numbers which are multiples of 2 and 5.
2. Telephone numbers within the same area code ending in 7 or beginning with 9.
3. Committees formed by two men and three women in a particular company (**Hint**: Think carefully about it).
4. People in the United States whose last name starts with C and ends with O.
5. Students in the Discrete Mathematics class who are taking the Data Structure class.
6. Passwords with seven characters that can be made using either letters from the English alphabet, or the digits from 1 through 9.
7. Seven – card hands from a standard 52-card deck having at least two threes.

**CARTESIAN PRODUCT**

We know that the sets \( A=\{1, 2\} \) and \( B=\{2, 1\} \) are the same. As we know, in the case of sets the order in which the elements are listed is not important. However, there are opportunities when order is important. When we want to state that 2 is the first element and 1 the second element, we talk about the ordered pair with first component 2 and second component 1. The ordered pair is denoted by parentheses instead of brackets as \((2,1)\). Therefore, \((1, 2) \neq (2, 1)\). However, \(\{1,2\}=\{2,1\}\).

The collection of all ordered pairs with the first element in the set \( A \) and the second element in the set \( B \) is denoted \( A \times B \).

If \( A=\{a,b\} \) and \( B=\{1, 2, 3\} \), \( A \times B=\{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\} \).

**EXERCISE 12**

Using the sets \( A \) and \( B \) from the previous example:

- Calculate \( B \times A \)
- Are \( A \times B \) and \( B \times A \) equal?
- Calculate \( A \times A \).

In a similar way, we can refer to ordered 3-ples.

\[(a, b, c) \neq (a, c, b) \neq (b, c, a) \neq (b, a, c)\]

The collection of triples with the first element in \( A \), second element in \( B \) and third element in \( C \) is denoted \( A \times B \times C \). Likewise, we can talk about 4-tuples, 5-tuples, etc.

**EXAMPLE**

Let’s consider the following table which corresponds to the departments-and-employees database of a small company.
### DEPT

<table>
<thead>
<tr>
<th>DEPT#</th>
<th>DNAME</th>
<th>BUDGET</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₁</td>
<td>Marketing</td>
<td>10 M</td>
</tr>
<tr>
<td>D₂</td>
<td>Development</td>
<td>12 M</td>
</tr>
<tr>
<td>D₃</td>
<td>Research</td>
<td>5 M</td>
</tr>
</tbody>
</table>

### EMP

<table>
<thead>
<tr>
<th>EMP#</th>
<th>ENAME</th>
<th>DEPT#</th>
<th>SALARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₁</td>
<td>Lopez</td>
<td>D₁</td>
<td>40 K</td>
</tr>
<tr>
<td>E₂</td>
<td>Cheng</td>
<td>D₁</td>
<td>42 K</td>
</tr>
<tr>
<td>E₃</td>
<td>Finzi</td>
<td>D₂</td>
<td>30 K</td>
</tr>
<tr>
<td>E₄</td>
<td>Soto</td>
<td>D₃</td>
<td>35 K</td>
</tr>
</tbody>
</table>

When we fix the heading for each table, each row in the DEPT table becomes a triple and each row in the EMP table becomes a 4-tuple. Using this language, the DEPT table is a set with three elements and the EMP table is a set with four elements.

What is the intersection of these two sets?

Now, consider the sets formed by the headings of the tables. Name them:

D={DEPT #, DNAME, BUDGET}, E={EMP#, ENAME, DEPT#, SALARY}

Notice that

\[
D \cap E = \{\text{DEPT#}\} \quad \text{and} \quad D \cup E = \{ \text{DEPT#, DNAME, BUDGET, EMP#, ENAME, SALARY} \}.
\]

In this case we can form a new table, called **THE JOIN OF DEPT AND EMP**. This table is shown below.

<table>
<thead>
<tr>
<th>DEPT #</th>
<th>DNAME</th>
<th>BUDGET</th>
<th>EMP#</th>
<th>ENAME</th>
<th>SALARY</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₁</td>
<td>Marketing</td>
<td>10 M</td>
<td>E₁</td>
<td>Lopez</td>
<td>40 K</td>
</tr>
<tr>
<td>D₁</td>
<td>Marketing</td>
<td>10 M</td>
<td>E₂</td>
<td>Cheng</td>
<td>42 K</td>
</tr>
<tr>
<td>D₂</td>
<td>Development</td>
<td>12 M</td>
<td>E₃</td>
<td>Finzi</td>
<td>30 K</td>
</tr>
<tr>
<td>D₃</td>
<td>Development</td>
<td>12 M</td>
<td>E₄</td>
<td>Soto</td>
<td>35 K</td>
</tr>
</tbody>
</table>

Observe that to make this new table, we just take the rows with non-empty intersection (think of each row as a set). For example, the first row of DEPT table and the first two rows of EMP table do not have empty intersection. This generates the first two rows of the join. On the other hand, the third row of DEPT table is disjoint with all the rows in EMP table. This is why D₃ does not appear in the join table.

**EXERCISE 13**

You are asked to create a database for the discrete mathematics class so that when the name of the student is entered all the courses the student has taken appear. This table can be interpreted as a subset of the Cartesian product between sets A and B.

1. Which are the sets A and B that you would consider?
2. Give an example of an element in AxB and one in BxA.

3. Give a situation for which we might want to consider a subset of BxA?

**COUNTING ELEMENTS OF THE UNION OR INTERSECTION**

When we know the number of elements of two sets and the number of elements of the intersection, we can determine the number of elements of the intersection. Let’s see this with an example. In a college there are 344 students who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?

Let A be the set formed by the calculus students and B the set formed by the discrete mathematics. What we need to find is |A ∪ B| by knowing |A|, |B| and |A ∩ B|.

From the diagram we obtain the number of elements of the union. It is the number of elements of A plus the number of elements in B, minus the number of elements in the intersection. We have to subtract the intersection once because we count twice (when we counted A and B). So, the follow holds

\[ |A ∪ B| = |A| + |B| - |A ∩ B| \]

This equality is also called the **Inclusion – Exclusion Principle** (you include and then exclude) in the case of two sets. Later, we will consider the most general case.

Using the inclusion-exclusion principle we can determine the number of students who are taking calculus or discrete mathematics as given above. We know that |A| = 344, |B| = 212, and |A ∩ B| = 188. Therefore, |A ∪ B| = 344 + 212 − 188 = 368.

Hence, there are 368 students who are taking both classes.

**EXERCISE 14**

Use the Inclusion-Exclusion Principle to solve each of the following problems. Indicate (label) clearly the sets you are considering.

1. Multiple personality disorder (MPD) is a condition in which different personalities exist within one person and at various times control that person’s behavior. In a recent survey of people with MPD, it was reported that “98% had been emotionally abused, 89% had been physically abused and **most** had experienced both types of abuse.” Make this statement more precise.
2. Among the 30 students registered for a course in discrete mathematics, 15 people know the PASCAL programming language, 12 know FORTRAN, and five know both of these languages.
   a) How many students know at least one of PASCAL or FORTRAN?
   b) How many students know only PASCAL?
   c) How many know only FORTRAN?
   d) How many know exactly one of the languages, PASCAL or FORTRAN?
   e) How many students know neither PASCAL, nor FORTRAN?