STRINGS

The license plate JKL 023S, the telephone number 992 4001, the password *52AamZP are examples of strings. A string is a sequence of characters. We are concerned about counting the number of strings. For example:

a. How many license plates can be made starting with three letters (upper case), followed by three characters that could be either numbers from the set \{1,2,\ldots,9\} or letters (upper case too)? Note that each of the characters comes from a specified set. Each of the sets where the characters come from is called an event. Each of the first three events is the set of letters from the English alphabet. Each of the last four events is the set formed by the union of letters from the English alphabet and the set of digits without zero.

b. How many passwords can be assigned using six characters that can be letters (upper or lower case) or numbers? Notice that there are six events and each of the events is the set formed by the union of natural number from zero to nine and letters from the English alphabet (upper and lower case).

c. What is the maximum number of different telephone numbers we can have within each area code? We make the assumption that a telephone number does not start with zero. The strings to consider are of the form 302 0018, 213 4099, … Each of these strings has seven characters, so we are dealing with strings of length seven. The second through the seventh characters come from the event \{0, 1, 2,\ldots, 9\} and the first character comes from the event \{1, 2, \ldots, 9\}.

d. There are many cases when we are interested in counting strings that consist of zeroes and ones. These are called bit strings (the term bit comes from binary digit). For instance 0010 and 1111 are 4-bit strings. By the way, a bit string of length eight is called a byte.

We will see how to count the number of possible strings like the ones mentioned in the examples above. Later on, more involved cases with strings will be considered.
EXAMPLE 1

Let’s count the number of strings of length two that can be made with the letters a and b.

Approach 1. Listing all possible strings:

aa, ab, ba, bb.

Approach 2. Generating the strings using a tree diagram.

What we do here is to place the choices for the first character in a vertical (or horizontal) arrangement. It is the first column corresponds to the first event. Now, after the first character is selected, we proceed to select the second character. Draw a line between the first and second characters. Notice that in front of each of the first characters we place all the elements from the second event. Observe that if we start with the leftmost characters and follow the lines toward the right, we obtain the same strings generated by the listing process mentioned in approach 1.

It is interesting to observe that when the listing approach is used, there must be a way to guarantee that all possible cases are considered. The tree diagram presents a more consistent approach to generate all the strings. Consequently we should be able to count all of them correctly. Of course, the drawback is that, very rapidly, the diagram can become very complicated if the events have large number of elements or there are many events to consider.

EXAMPLE 2

Let’s count all the strings of length three where the first and last characters can be any of the letters a or b, and the second character is either 1, 2, or 3.
**Approach 1**: Listing of all the strings:

\[ a1a, a1b, a2a, a2b, a3a, a3b, b1a, b1b, b2a, b2b, b3a, b3b \]

**Approach 2**: Using a tree diagram as the one shown below. If we follow each of the paths from the leftmost to the rightmost characters, a complete list of all the strings is obtained. Observe once again that the first column corresponds to the first event. In front of each of the elements from event one there is a complete list of elements from event two. Finally, in front of each of the listed elements in column two there is a complete list of elements from event three. Following the lines from left to right you will find that these correspond to the same strings listed in approach 1.

**EXERCISE 1**
Produce the tree diagram to determine the number of strings of length three we can make with the characters 1, 2, a, b, c, where the first and third characters are numbers and the second one is a letter.
Counting strings by listing may be a very cumbersome process. Let’s make some observations to be able to count strings without listing them all.

In the case of the strings of length two, think of them as a sequence of two events. Let’s represent the events by these two slots.

The first slot (corresponding to the first event) can be occupied by either a or b. If the selected character is \( \text{a} \), we are dealing with all the strings of the form

\[
\text{a} \quad \_ \quad \_ 
\]

Now, for the second slot (corresponding to the second event) there are two choices a or b. So, for the first slot there are two choices. For each of the two possible choices that existed for the first slot, there are two choices for the second slot. Thus, the total number of different strings is

\[2 \times 2 = 2^2.\]

For strings of length three with first and third characters a or b, and second character 1, 2, 3, follow a similar reasoning as we did for strings of length two. For the first slot (first
event) there are two choices either a or b. If we select b for the first slot, then we are counting all the strings of the form

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   b
     ___ ___ ___
```

Now, for the second slot there are three choices: 1, 2, or 3. These are all possible strings after selecting b for the first character and all possible second characters:

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   b 1 ___   b 2 ___   b 3 ___
```

The third slot can be either a or b. Observe that the first event has two elements (number of ways to select the first character), after it is selected there are three choices for the second event (number of ways to select the second character) and finally there are two choices for the last event for a total of

\[
\text{Number of ways to select the first character} = \text{The number of elements of the first event.}
\]

\[
\text{Number of ways to select the second character} = \text{The number of elements of the second event.}
\]

\[
\text{Number of ways to select the third character} = \text{The number of elements of the third event.}
\]

\[
2 \times 3 \times 2 = 12 \text{ different strings.}
\]
EXERCISE 2

1. How many strings of length three are there such that the first slot can be either 1 or 2, the second slot can be either m or M and the third slot can be any lower case letter of the English alphabet?

2. How many strings of length five are there where each entry can be either a zero or a one?

3. How many strings of length four can you make with zeroes and ones, knowing that the first entry is a one and the last entry is a zero?

4. How many strings of length 25 are there with just zeroes and ones?

5. Knowing that a telephone number does not begin with a zero, how many different telephone numbers can be in each area code? (HINT: We still have seven-digit telephone numbers!)

6. At an elementary school the teacher asked the students to line up in a way that two people of the same sex are not consecutive in line. If there are five boys and five girls, in how many different ways can the line be formed? Solve this using the string approach, without producing a tree-diagram.

MULTIPLICATION PRINCIPLE

Consider a sequence of events $E_1, E_2, E_3, \ldots, E_n$. If after event $E_1$ happens event $E_2$ happens, and after event $E_2$ happens event $E_3$ happens and so on until the last event happens, the sequence of events $E_1, E_2, E_3, \ldots, E_n$ happens in $|E_1| |E_2| |E_3| \ldots |E_n|$ ways.

Counting the number of strings considered in the examples before is an application of this principle.

HOW TO COUNT SUBSETS USING STRINGS

Now we want to count the number of subsets of a given set by translating the problem into a bit string problem (a string with 0’s and 1’s).

To this end, let’s consider the set $A = \{a, b, c\}$. A subset of $A$ is to be represented by a bit string of length three (there are three elements in $A$). Think of the slots as labeled by
the elements in the set, so we can keep track whether or not the elements are in the subset. *Placing 1 in the first slot indicates that a is in the subset and placing 0 indicates that a is not an element of the subset.* A one or zero in the second slot indicates whether or not b is in the subset, and likewise with the third slot. In this way, we identify 3-bit strings with subsets of a set with three elements. A unique bit string represents each subset and each bit string represents a unique subset. This correspondence is shown in the table below.

<table>
<thead>
<tr>
<th>Subsets of {a, b, c}</th>
<th>Strings</th>
</tr>
</thead>
<tbody>
<tr>
<td>a b c</td>
<td>a b c</td>
</tr>
<tr>
<td>∅</td>
<td>0 0 0</td>
</tr>
<tr>
<td>{a}</td>
<td>1 0 0</td>
</tr>
<tr>
<td>{b}</td>
<td>0 1 0</td>
</tr>
<tr>
<td>{c}</td>
<td>0 0 1</td>
</tr>
<tr>
<td>{a, b}</td>
<td>1 1 0</td>
</tr>
<tr>
<td>{a, c}</td>
<td>1 0 1</td>
</tr>
<tr>
<td>{b, c}</td>
<td>0 1 1</td>
</tr>
<tr>
<td>{a, b, c}</td>
<td>1 1 1</td>
</tr>
</tbody>
</table>

Thus, if we want to count the number of subsets of a set with three elements, it is the same as counting the number of 3-bit strings, which is $2^3$.

*Note:* Even though in a set the order of the elements is not important, when we associate bit strings with subsets, the position of the elements is fixed for the discussion (they become the labels of the slots of the string).

**EXERCISE 3**

How many bit strings are there of length n? Do it for n=1, 2, ..., 5.
FACT

Since each bit string uniquely represents a subset and vice versa, the number of subsets of a set with \( n \) elements is the same as the number of bit strings of length \( n \), which is \( 2^n \).

EXERCISE 4

John, Sally, Sam, Lucas and Rupert are the candidates to form a committee. We consider the candidates as a set.

1. In front of each choice of committee members, write the corresponding string with zeroes and ones that represents it.

   \{John, Sam, Lucas\} ↔

   \{Sally, Sam, Lucas, Rupert\} ↔

   \{Sally, Rupert\} ↔

2. In front of each string with zeroes and ones, write the committee that this represents

   01001 ↔

   11111 ↔

   00000 ↔

   10101 ↔

3. In how many ways can we choose a committee of four people? How about a committee of two people?

EXERCISE 5

1. How many subsets does a set with 10 elements have?

2. Calculate \(|\mathcal{P}(A)|\) for a set \( A \) with \(|A| = 5\)
3. In the case of the experiments to be taken into the space we know that there are $2^{12}$ choices of experiments to take. Use bit strings to answer the questions below:
   a. How many different ways are there to choose three experiments?
   b. How about seven experiments?