Laboratory I.10
It All Adds Up

Goals

- The student will strengthen his/her knowledge of Riemann sums and evaluate them using Derive.
- The student will see applications of integrals as accumulations of changes.
- The student will review curve fitting skills.

Before the Lab

The “Before the Lab” and “Ready for Lab” sections are longer than usual for this lab; be sure to start early. (On the other hand, the “In the Lab” and “After the Lab” are correspondingly shorter.)

Part I: The Initial Scenario

The Environmental Protection Agency recently investigated a spill of radioactive iodine. Measurements showed the ambient radiation levels at the site to be four times the maximum acceptable limit of 0.6 millirems/hour (abbreviated mrems/hr). The EPA ordered an evacuation of the area. You will investigate several aspects of this incident using integrals and other basic information about functions.

At the beginning of the investigation, the emission rate was measured to be 2.4 mrems/hr. Over the first four hours, the following data were recorded:

<table>
<thead>
<tr>
<th>Time (hrs)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation (mrems/hr)</td>
<td>2.4</td>
<td>2.39041</td>
<td>2.38087</td>
<td>2.37137</td>
<td>2.36190</td>
</tr>
</tbody>
</table>

There are some questions on this scenario in the “Ready for Lab?” section.

Part II: Riemann Sums in Derive

By now in class, you’ve been introduced to Riemann sums, whose limit is the definite integral. For example, in the box in the text on p. 154, we have:

\[
\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i) \Delta t \quad (\text{left hand sum})
\]

\[
\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i) \Delta t \quad (\text{right hand sum})
\]

Now, the right hand side of each of the equations above is a limit as \( n \) goes to infinity, so the bigger \( n \) gets, the closer the Riemann sum is to the exact value of the integral on the left hand side. The purpose of this “Before the Lab” section is to work through the translation of the
algebraic summation notation \[ \sum_{i=0}^{n-1} f(x_i) \Delta x \text{ and } \sum_{i=1}^{n} f(x_i) \Delta x \] into Derive’s version of the same thing. To calculate a Riemann sum we (and Derive) need to know four things: what’s the function \( f \)? what are the endpoints \( a \) and \( b \)? and what’s \( n \), the number of terms in the sum?

We first need to re-write \( x_i \) and \( \Delta x \) in terms of \( a \), \( b \) and \( n \). \( \Delta x \) is the distance from \( a \) to \( b \), cut into \( n \) pieces, so that’s \( \Delta x = \frac{b - a}{n} \). To understand \( x_i \), let’s look at the pattern we see:

\[
\begin{align*}
x_0 &= a \\
x_1 &= a + \Delta x = a + \frac{b - a}{n} \\
x_2 &= x_1 + \Delta x = a + 2 \Delta x = a + 2 \frac{b - a}{n} \\
x_3 &= x_2 + \Delta x = a + 3 \Delta x = a + 3 \frac{b - a}{n} \\
&\text{etc.}
\end{align*}
\]

See the pattern? \( x_i = a + i \frac{b - a}{n} \). Substituting our new formulas for \( x_i \) and \( \Delta x \) into the right hand sum formula, we get:

\[
\sum_{i=1}^{n} f\left(a + i \frac{b - a}{n}\right) \frac{b - a}{n}
\]

Our goal was to write the Riemann sum just in terms of \( f \), \( a \), \( b \), and \( n \), and we have. When you get to the lab, you can type in the above Riemann sum in two lines:

\begin{verbatim}
Author f(x) :=
then
Author RRS(a, b, n) := sum(f(a + i(b - a)/n)(b - a)/n, i, 1, n)
\end{verbatim}

The first line, \( f(x) := \), tells Derive that the letter \( f \) will stand for a function of \( x \) in the next formula. The second line creates a function called \( \text{RRS} \) (= Right Riemann Sum) which, given the necessary ingredients of \( f \), \( a \), \( b \), and \( n \), creates the exact sum we see above.

Suppose we wanted to use this in lab to estimate \( \int_{2}^{5} x^3 \, dx \) with \( n = 7 \) rectangles. Then, after doing the above, we would:

\begin{verbatim}
Author f(x) := x^3
Author RRS(2, 5, 7)
Approximate the result.
\end{verbatim}

If you try this in lab, you should get 178.29.

So, there’s a “Ready for Lab?” question for this section as well.
In the Lab

0. Author \( R(t) := \) your formula for the iodine data, as well as the RRS and LRS formulas. Be sure to use \( R \), instead of \( f \), in your RRS and LRS formulas, since if you use \( f \), Derive will look for a function called \( f \) and not find it.

1. a) Use RRS and LRS with \( n = 4 \) to estimate the integral \( \int_{0}^{4} R(t) \, dt \) from \( t = 0 \) to \( t = 4 \). Fill in the appropriate lines in Table 1. Since you know that the exact integral is somewhere between the two, estimate the maximum error you could make if you used the average of the two estimates to evaluate the integral.

b) We know from class that increasing \( n \) increases the accuracy of the estimate you get from Riemann sums. Continue filling out lines on the table--you won’t have to use them all--until your estimated maximum error is no more than 0.0025.

c) Use Derive’s \texttt{Calculus Integrate} command, followed by \texttt{Approximate}, to evaluate \( \int_{0}^{4} R(t) \, dt \) exactly. (In the \texttt{Integrate} dialog box, you can use \( R(t) \) for the function; you’re doing a definite integral, and the upper and lower bounds are 4 and 0 respectively.) Fill in the blank below Table 1. Were you right to stop where you did in the table? That is, is the average of your last row within 0.0025 of the exact value?

2. a) In “Ready for Lab?” #4, you found the time when the iodine radiation levels return to the “acceptable” level of 0.6 mrem/hr. Call that time \( t_a \). Use \texttt{Calculus Integrate} to evaluate \( \int_{0}^{t_a} R(t) \, dt \).

   (Of course, when you enter this into Derive, you’ll use your actual value for \( t_a \), not the symbols “\( t_a \)”.) Fill in the appropriate spot in Table 2. You will be asked about the meaning of this integral in an “After the Lab” question.

b) Continue filling in rows of Table 2--again, you won’t use them all--until the value you get is less than 1. Then add all the integrals you found and fill in the blank below Table 2. In the “After the Lab” section again, you will be asked about the meaning of the sum.

c) Finally, you can, for some integrals, you can use infinity (\( \infty \)) as a bound. Use \texttt{Calculus Integrate} to evaluate \( \int_{0}^{4} R(t) \, dt \). (Click on the \( \infty \) key in the \texttt{Integrate} dialog box when you are entering the upper bound.)
### Table 1

<table>
<thead>
<tr>
<th>n</th>
<th>LRS</th>
<th>RRS</th>
<th>Max error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Exact value of $\int_0^4 R(t) \, dt$ from Calculus Integrate

### Table 2

<table>
<thead>
<tr>
<th>Bounds on the integral</th>
<th>Value of the integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to $t_a$</td>
<td></td>
</tr>
<tr>
<td>$t_a$ to $2t_a$</td>
<td></td>
</tr>
<tr>
<td>$2t_a$ to $3t_a$</td>
<td></td>
</tr>
<tr>
<td>$3t_a$ to $4t_a$</td>
<td></td>
</tr>
<tr>
<td>$4t_a$ to $5t_a$</td>
<td></td>
</tr>
<tr>
<td>$5t_a$ to $6t_a$</td>
<td></td>
</tr>
<tr>
<td>$6t_a$ to $7t_a$</td>
<td></td>
</tr>
<tr>
<td>$7t_a$ to $8t_a$</td>
<td></td>
</tr>
</tbody>
</table>

Sum of integrals above

\[
\int_0^4 R(t) \, dt = \]

\[
\int_0^4 R(t) \, dt =
\]
Ready for Lab?

Questions on the initial scenario:

1. Based on the numbers in the table, calculate an upper and lower bound for the total amount of radiation (in mrems) emitted in the first four hours.

2. Let \( R(t) \) denote the radiation level in mrems/hr at time \( t \) hours after the iodine spill. Write an integral that represents the total amount of radiation (in mrems) emitted in the first four hours. (Warning: it’s easier than it looks.)

3. If the EPA relied solely on measurements, it would have to keep measuring every hour to see when it was safe to return to the site of the spill. It would be better if there was a formula which predicted radiation levels at future times. There is!. Radiation levels decay exponentially, so \( R(t) \) can be written as \( R_0 e^{kt} \). Find \( R_0 \) and \( k \) for the data in the table.

4. Based on your formula from #3, when will the radiation level fall to the EPA’s estimate of an “acceptable” radiation level of 0.6 mrems/hr?

5. Based on the data in the Table for radiation, is the left Riemann sum going to be an underestimate, or an overestimate? Why? is the right Riemann sum going to be an underestimate, or an overestimate? Why?
**Question on Riemann Sums in Derive:**

6. Write down a Derive formula for the Left Riemann Sum, based on what was in the Riemann Sum section of “Before the Lab”.

**After the Lab**

1. From Table 1, by what factor does doubling the number of rectangles $n$ seem to reduce the possible error?

2. What is the meaning of the number you calculate in each of parts (a), (b), and (c) of “In the Lab” #2?